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The non-isospectral AKNS hierarchy with reality conditions restriction

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Abstract
In this paper, we will prove the existence of the non-isospectral AKNS hierarchy with reality conditions restriction and construct the matrix form Darboux transformation. Using this Darboux transformation, the solutions of the relevant nonlinear equations can be expressed explicitly.

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1. Introduction

The AKNS hierarchy is one of the most important integrable systems, which was first introduced by Ablowitz *et al* [1]. Many important nonlinear differential equations can be equivalent to the integrability condition of the AKNS hierarchy [12, 13, 23]. The Darboux matrix formalism was introduced in connection with the dressing method as originated by Zakharov and Shabat in [28] and described in the text [27]. Important work on the Darboux matrix method was conducted by Matveev and Salle [21], Neugebauer and Meinel [22], Levi [17, 18], and Gu [8–15]. One may obtain nontrivial solutions to the integrable system and the relevant nonlinear PDE by using Darboux transformation on a seed solution.

The non-isospectral problem is the problem in which the spectral parameter depends on the time or space variables. The Ernst equations [6, 7], which are the Einstein equations for the axially symmetric gravitational fields, were represented as the zero-curvature equations of non-isospectral problems by Maison [20]. An excellent account of the Darboux matrix method paying particular attention to non-isospectral problems has been given by Harrison [16], Kramer and Neugebauer [19], and Cieśliński [3–5].

In this paper, we focus on the non-isospectral AKNS hierarchy with reality conditions restriction. In section 2, one may review the main conclusions on the non-isospectral AKNS hierarchy and its Darboux transformation, which are collected in the literature [30] and will be used in the following sections. In section 3, we will show the existence of the non-isospectral

AKNS hierarchy with reality conditions restriction, and construct its matrix form Darboux transformation. In section 4, one may see its application to the non-isospectral su (2)-hierarchy, whose integrability condition is equivalent to the non-autonomous NLS equation.

2. The non-isospectral AKNS hierarchy and its Darboux transformation

We assume throughout that the matrix $J = \mathrm{diag}(J_1, \ldots, J_N) \in \mathrm{sl}(N, \mathbb{C})$ in this paper is fixed, diagonal, with distinct complex eigenvalues, and

$$\mathrm{sl}(N,\mathbb{C})_J = \{X \in \mathrm{sl}(N,\mathbb{C}) : [J, X] = 0\} \tag{2.1}$$

$$\mathrm{sl}(N,\mathbb{C})_J^{\perp} = \{X \in \mathrm{sl}(N,\mathbb{C}) : \mathrm{tr}(YX) = 0 \quad \text{for all} \quad Y \in \mathrm{sl}(N,\mathbb{C})_J\} \tag{2.2}$$

denote the centralizer of J and its orthogonal complement in $\mathrm{sl}(N,\mathbb{C})$ respectively. One can easily verify the following facts:

Lemma 2.1. $\mathrm{sl}(N,\mathbb{C})$ *has the direct sum decomposition* $\mathrm{sl}(N,\mathbb{C}) = \mathrm{sl}(N,\mathbb{C})_J \oplus \mathrm{sl}(N,\mathbb{C})_J^{\perp}$ *with respect to vector space. The mapping* $\mathrm{ad}J : \mathrm{sl}(N,\mathbb{C}) \to \mathrm{sl}(N,\mathbb{C})_J^{\perp}$ *is a homomorphism, and* $\ker(\mathrm{ad}J) = \mathrm{sl}(N,\mathbb{C})_J$*, which is equivalent to the fact that the restriction of the mapping* $\mathrm{ad}J : \mathrm{sl}(N,\mathbb{C})_J^{\perp} \to \mathrm{sl}(N,\mathbb{C})_J^{\perp}$ *is an isomorphism.*

The non-isospectral AKNS hierarchy is the linear system of differential equations

$$\begin{cases} \Phi_x = U(\lambda)\Phi = (\lambda J + P(t,x))\Phi \\ \Phi_t = V(\lambda)\Phi = \sum_{i=0}^{n} V_i(t,x)\lambda^i \Phi \end{cases} \tag{2.3}$$

where $P \in \mathrm{sl}(N,\mathbb{C})_J^{\perp}$ *and* $V_i \in \mathrm{sl}(N,\mathbb{C})$ are matrices and the spectral parameter λ satisfies the scalar equation

$$\lambda_t = \sum_{i=0}^{n} f_i \lambda^i. \tag{2.4}$$

The system is integrable if and only if the zero curvature condition

$$U_t - V_x + [U, V] = 0 \tag{2.5}$$

holds. If the coefficients f_i in (2.4) vanish, then the spectral parameter λ is independent of t, and the system (2.3) degenerates to the standard AKNS system.

For the standard integrable AKNS hierarchy, $V_i(t,x)$ is determined uniformly by $P(t,x)$ up to the integral constants $\alpha_i(t)$ (see [13]), and the jth flow is a differential equation. The non-isospectral AKNS hierarchy is much more different. To determine $V_i(t,x)$ in the non-isospectral AKNS hierarchy, $P(t,x)$ should be in some Schwartz space (see [30]), while the jth flow of the non-isospectral AKNS hierarchy is an integral-differential equation. We state the conclusion as the following proposition:

Theorem 2.2 (From [30]). *Assume that* $P(t,\cdot) \in \mathscr{S}\left(\mathbb{R}, \mathrm{sl}(N,\mathbb{C})_J^{\perp}\right)$*, i.e.*

$$\lim_{x\to\infty} |x|^k \partial_x^m (P(t,x)) = 0 \tag{2.6}$$

for all t and non-negative integers k, m. Set

$$V_i^{\mathrm{diag}} = \pi_0(V_i), \qquad V_i^{\mathrm{off}} = \pi_1(V_i), \tag{2.7}$$

where $\pi_0, \pi_1 \in \mathrm{End}(\mathrm{sl}(N, \mathbb{C}))$ *denote the projections of* $\mathrm{sl}(N, \mathbb{C})$ *onto* $\mathrm{sl}(N, \mathbb{C})_J$ *and* $\mathrm{sl}(N, \mathbb{C})_J^{\perp}$ *respectively. Then* V_i *is determined uniformly by* P *up to the integral constants* $\alpha_i(t) \in \mathrm{sl}(N, \mathbb{C})$*), and the recurrence formulae are*

$$V_n^{\mathrm{off}} = 0, \tag{2.8}$$

$$V_i^{\mathrm{diag}} = \alpha_i(t) + f_i J x + \int_{-\infty}^{x} \pi_0\big(\big[P, V_i^{\mathrm{off}}\big]\big)\,\mathrm{d}x, \qquad (0 \leqslant i \leqslant n-1), \tag{2.9}$$

$$V_i^{\mathrm{off}} = (\mathrm{ad}\,J)^{-1}\big(V_{i+1,x}^{\mathrm{off}} - \pi_1([P, V_{i+1}])\big). \tag{2.10}$$

The zero curvature equation of the hierarchy is equivalent to the nonlinear equation

$$P_t - V_{0,x}^{\mathrm{off}} + \big[P, V_0^{\mathrm{diag}}\big] = 0. \tag{2.11}$$

Moreover, for all t and non-negative integers k, m,

$$\lim_{x\to-\infty} |x|^k \partial_x^m (V_i - f_i J x - \alpha_i(t)) = 0 \qquad (0 \leqslant i \leqslant n). \tag{2.12}$$

The Darboux transformation for the non-isospectral AKNS hierarchy may be constructed in the following way (see [29, 30] for the proof). Without any loss of generality, we assume that the eigenvalues of J satisfy $\mathrm{Re}\,J_1 > \mathrm{Re}\,J_2 > \cdots \mathrm{Re}\,J_N$. Let $\Phi(t, x, \lambda)$ denote the elementary solution of the system (2.3), $C \in \mathrm{GL}(N, \mathbb{C}^N)$ be a constant matrix, and choose the spectral parameters

$$\lambda_1(t) = \cdots = \lambda_{k_0}(t) < 0 < \lambda_{k_0+1}(t) = \cdots = \lambda_N(t), \qquad \text{for} \quad t \in (-t_0, t_0)$$

satisfy (2.4). Set

$$D(\lambda) = p(\lambda)(\lambda I - H\Lambda H^{-1}),$$

where

$$\begin{aligned} &\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N),\\ &H = (\Phi_1(t, x, \lambda_1) col_1 C, \ldots, \Phi_N(t, x, \lambda_N) col_N C),\\ &p(\lambda)^{-N} = \det P(\lambda) = (\lambda - \lambda_1)\cdots(\lambda - \lambda_N) \end{aligned} \tag{2.13}$$

then one may find that $D(\lambda)$ is the Darboux transformation matrix for the AKNS hierarchy. It is equivalent to saying that such

$$U'(\lambda) = DU(\lambda)D^{-1} + D_x D^{-1}, \qquad V'(\lambda) = DV(\lambda)D^{-1} + D_t D^{-1} \tag{2.14}$$

preserve the same polynomial structure as $U(\lambda)$ and $V(\lambda)$, and

$$P' = P + [J, S] \in \mathscr{S}(\mathbb{R}, \mathrm{sl}(N, \mathbb{C})_J^{\perp}) \tag{2.15}$$

holds for any fixed $t \in (-t_0, t_0)$. Moreover, the integral constants related to U' and V' are determined by the following:

$$\alpha_i'(t) = \alpha_i(t) + \sum_{k=0}^{n-j+1} f_{j+k+1}\Lambda^k - g_j \qquad (0 \leqslant j \leqslant n), \tag{2.16}$$

where g_j is defined as the coefficient of the polynomial

$$g(\lambda) = \frac{1}{N}\sum_{i=1}^{N} \frac{f(\lambda) - f(\lambda_i)}{\lambda - \lambda_i} = \sum_{i=0}^{n+1} g_i \lambda^i. \tag{2.17}$$

For convenience, we denote the last two terms of the right-hand side of (2.16) by $\beta_i(t)$. If $\beta_i(t)$ vanishes for each i, then $D(\lambda)$ is an auto-Bäcklund transformation, otherwise it is not.

Noting that β_i is determined by Λ, which is independent of $U(\lambda)$ and $V(\lambda)$, so one may get the nontrivial solution to the hierarchy with the integral constants α_i by acting the Darboux transformation on the hierarchy with the integral constants $\alpha_i - \beta_i$.

Remark 2.1. If only $\beta_0(t) \neq 0$, one may define a gauge matrix $G(t)$ to attain an auto-Bäcklund transformation by setting $\tilde{D}(\lambda) = G(t)D(\lambda)$, where $G(t)$ solves the differential equation $G_t = \beta_0(t)G$.

3. Reality conditions

To reduce the zero curvature condition, we need to impose reality conditions on the AKNS hierarchy. We explain reality conditions given by involutions of $\mathrm{sl}(N, \mathbb{C})$.

Definition 3.1. *Let $\mathcal{G}$ denote a real form of $\mathrm{sl}(N, \mathbb{C})$; i.e $\mathcal{G}$ is the fixed-point set of some complex conjugate linear Lie algebra involution σ of $\mathrm{sl}(N, \mathbb{C})$. A map U from $\mathbb{C}$ to $\mathrm{sl}(N, \mathbb{C})$ is said to satisfy the $\mathcal{G}$-reality condition if $\sigma(V(\bar{\lambda})) = V(\lambda)$ for all $\lambda \in \mathbb{C}$. Moreover, if τ is a complex linear Lie algebra involution of $\mathrm{sl}(N, \mathrm{C})$ such that $\sigma\tau = \tau\sigma$ and $\tau(U(-\lambda)) = U(\lambda)$, we say that $V(\lambda)$ satisfies the $\mathcal{G}$-reality condition twisted by τ.*

There are some typical examples.

(i) Set $\sigma(A) = -A^\dagger$, then $\mathcal{G} = \mathrm{su}(N)$. $V(\lambda)$ satisfies the $\mathrm{su}(N)$-reality condition if $V(\bar{\lambda})^\dagger + V(\lambda) = 0$ for all $\lambda \in \mathbb{C}$.

(ii) Set $\sigma(A) = -MA^\dagger M$, where $M = \mathrm{diag}(I_k, -I_{N-k})$, then $\mathcal{G} = \mathrm{su}(k, N-k)$. $V(\lambda)$ satisfies the $\mathrm{su}(k, N-k)$-reality condition if $V(\bar{\lambda})^\dagger M + MV(\lambda) = 0$ for all $\lambda \in \mathbb{C}$.

(iii) Set $\sigma(A) = \bar{A}$ and $\tau(A) = -A^T$, then $\mathcal{G} = \mathrm{sl}(N, \mathrm{R})$. $V(\lambda)$ satisfies the $\mathrm{sl}(N, \mathrm{R})$-reality condition twisted by τ if $\overline{V(\bar{\lambda})} = V(\lambda)$ and $V(-\lambda)^T + V(\lambda) = 0$.

It is clear that $V(\lambda) = \sum_i V_i\lambda^i$ satisfies the reality condition if and only if V_i satisfies the reality condition for each i. For the twisted reality condition, let $\mathcal{K}$ and $\mathcal{K}'$ denote the ± 1 eigenspaces of τ on $\mathcal{G}$; a direct computation shows that $V(\lambda) = \sum_i V_i\lambda^i$ satisfies the $\mathcal{G}$-reality condition twisted by τ if $V_{2k} \in \mathcal{K}$ and $V_{2k+1} \in \mathcal{K}'$ for all k.

To reduce the zero-curvature condition, we require both $U(\lambda)$ and $V(\lambda)$ satisfying the same reality condition. However, $V(\lambda)$ is determined by $U(\lambda)$ and the integral constants $\alpha_i(t)$, so we need to prove that the $\mathcal{G}$-reality condition on $U(\lambda)$ implies the $\mathcal{G}$-reality condition on $V(\lambda)$. Noting that

$$\sigma\, \mathrm{ad}(J)(A) = \sigma[J, A] = [\sigma(J), \sigma(A)] = [J, \sigma(A)] = \mathrm{ad}(J)\sigma(A) \tag{3.1}$$

holds for all $J, A \in \mathcal{G}$, one can prove the following proposition easily.

Proposition 3.1. *If $U(\lambda)$ satisfies the $\mathcal{G}$-reality condition, the integral constant $\alpha_i(t) \in \mathcal{G}$ for all i, and the polynomial $f(\lambda)$ has the real coefficients, then $V(\lambda)$ satisfies the $\mathcal{G}$-reality condition.*

For the case of twisted reality condition, we also have the similar proposition.

Proposition 3.2. *Let $\sigma, \tau, \mathcal{G}, \mathcal{K}, \mathcal{K}'$ and $U(\lambda), V(\lambda)$ be as above. $U(\lambda)$ satisfies the $\mathcal{G}$-reality condition twisted by τ, the integral constants $\alpha_{2k}(t) \in \mathcal{K}, \alpha_{2k+1} \in \mathcal{K}'$ for all k, and the polynomial $f(\lambda)$ with the real coefficients satisfies $f(\lambda) + f(-\lambda) = 0$, then $V(\lambda)$ satisfies the $\mathcal{G}$-reality condition twisted by τ.*

The Darboux transformation for the $\mathcal{G}$-hierarchy is required to preserve the reality condition additionally, and it may be realized by choosing special Λ. We list some typical cases below (see [29] for the proof):

(i) To preserve the su(N)-reality condition, one may choose $\Lambda = \mathrm{diag}(\lambda_0, \ldots, \lambda_0, \bar{\lambda}_0, \ldots, \bar{\lambda}_0)$.

(ii) To preserve the sl($N, \mathbb{R}$)-reality condition twisted by τ, one may choose $\Lambda = \mathrm{diag}(\lambda_0, \ldots, \lambda_0, -\lambda_0, \ldots, -\lambda_0)$.

4. Non-isospectral su(N)-hierarchy and non-autonomous NSL equation

Now we use the Darboux transformation to get the soliton solution of non-autonomous NLS equation, which is the second flow of a non-isospectral su(2)-hierarchy. Set $f(\lambda) = 2\lambda^2$, $J = \mathrm{diag}(\mathrm{i}, -\mathrm{i})$, $\alpha_2 = -2J$, $\alpha_1 = \alpha_0 = 0$ and

$$P = \begin{pmatrix} 0 & \mathrm{i}u(t,x) \\ -\mathrm{i}u(t,x) & 0 \end{pmatrix},$$

then the entries a_0, b_0, c_0 in $V_0(\lambda)$ are followed from (2.9) and (2.10) as

$$b_0 = -\bar{c}_0 = -(u_x - xu_x - u), \qquad a_0 = -|u|^2 + x|u|^2 + \partial^{-1}|u|^2, \tag{4.1}$$

and according to (2.11) the zero-curvature equation is equivalent to

$$\mathrm{i}u_t = (u_{xx} - 2u_x - xu_{xx}) + 2|u|^2u - 2x|u|^2u - 2u\partial^{-1}|u|^2, \tag{4.2}$$

which is called the non-autonomous NLS equation. It is easy to find $\lambda = -\frac{2t+\kappa\mathrm{i}}{\kappa^2+4t^2}$, $\kappa \in \mathbb{C}$ solving (2.4). Set $u = 0$ as the seed solution of (4.2) and solve the relevant system (2.3),

$$\begin{cases} \Phi_x = \lambda J\Phi = \begin{pmatrix} \mathrm{i}\lambda & 0 \\ 0 & -\mathrm{i}\lambda \end{pmatrix}\Phi, \\ \Phi_t = V_2\Phi = \begin{pmatrix} 2\mathrm{i}(x-1)\lambda^2 & 0 \\ 0 & -2\mathrm{i}(x-1)\lambda^2 \end{pmatrix}\Phi, \end{cases} \tag{4.3}$$

then we attain the elementary solution

$$\Phi(x,t,\lambda) = \begin{pmatrix} \exp\mathrm{i}(\lambda x - \lambda) & 0 \\ 0 & \exp\mathrm{i}(-\lambda x + \lambda) \end{pmatrix}. \tag{4.4}$$

Noting that the eigenvalues of J are pure imaginary, we may set $\kappa_0 > 0$ and

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & \bar{\lambda}_0 \end{pmatrix} = -\frac{1}{\kappa_0^2 + 4t^2}\begin{pmatrix} 2t + \kappa_0\mathrm{i} & 0 \\ 0 & 2t - \kappa_0\mathrm{i} \end{pmatrix}, \tag{4.5}$$

to preserve the asymptotic condition (2.15) and su(N)-reality condition, then it follows from (2.16) that

$$\beta_2(t) = \beta_1(t) = 0, \qquad \beta_0 = 2\mathrm{i}(\mathrm{Im}\,\Lambda_0).$$

Denote $\xi = \exp\mathrm{i}(\lambda_0 x - \lambda_0)$ and choose

$$H = (\Phi(t,x,\lambda_0)(1,1)^T, \qquad \Phi(t,x,\bar{\lambda}_0)(-1,1)^T) = \begin{pmatrix} \xi & -\bar{\xi}^{-1} \\ \xi^{-1} & \bar{\xi} \end{pmatrix}, \tag{4.6}$$

then a direct calculation shows

$$S = H\Lambda H^{-1} = \frac{1}{1+|\xi|^4}\begin{pmatrix} \lambda_0|\xi|^4 + \bar{\lambda}_0 & 2\xi^2\,\mathrm{Im}\,\lambda_0 \\ 2\bar{\xi}^2\,\mathrm{Im}\,\lambda_0 & \lambda_0 + \bar{\lambda}_0|\xi|^4 \end{pmatrix}. \tag{4.7}$$

Note that only $\beta_0(t) \neq 0$, then we may set

$$G(t) = \exp\left(2\mathrm{i}\int_0^t (\mathrm{Im}\,\Lambda)\,\mathrm{d}t\right) \in \mathrm{SU}(2),$$

so that the tranformation

$$D'(\lambda) = G(t)D(\lambda) = G(t)p(\lambda)(\lambda I - H\Lambda H^{-1}) \tag{4.8}$$

preserves the su(N)-reality condition, hence it is an auto-Bäcklund transformation. According to (2.15), the single-soliton solution to the non-autonomous NLS equation can be expressed as

$$\begin{aligned} u(t,x) &= \frac{4\xi^2 \operatorname{Im}\lambda_0}{1+|\xi|^4} \exp\left(4\mathrm{i}\int_0^t (\operatorname{Im}\lambda_0)\,\mathrm{d}t\right) \\ &= \frac{4\xi^2 \operatorname{Im}\lambda_0}{1+|\xi|^4}\left(\frac{\kappa_0^2 - 4t^2}{\kappa_0^2 + 4t^2} + \frac{4t\kappa_0 i}{\kappa_0^2+4t^2}\right). \end{aligned} \tag{4.9}$$

Set the single solution as the seed solution; the 2-soliton solution can also be attained by this means.

Remark 4.1. This method may also be used to derive a Darboux transformation matrix for the twisted sl$(N,\mathbb{R})$-hierarchy. One can find an application in [30].

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